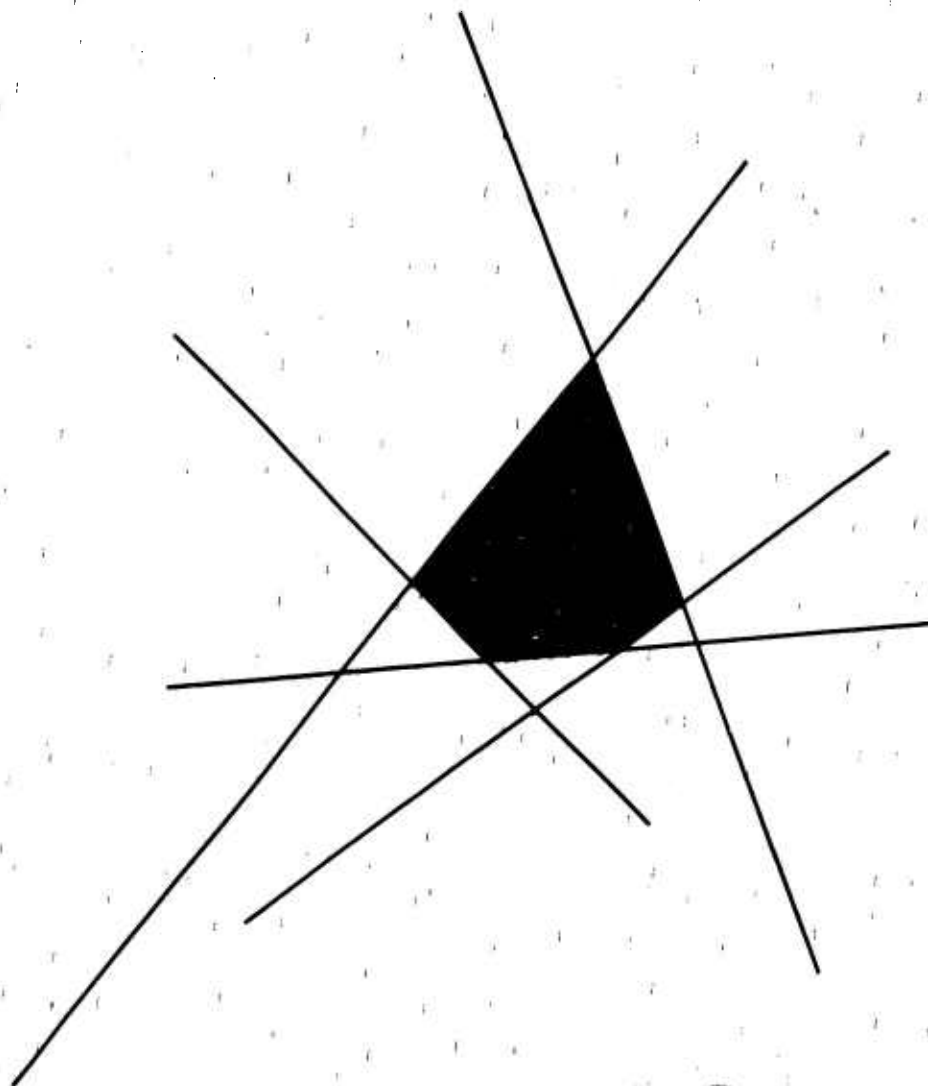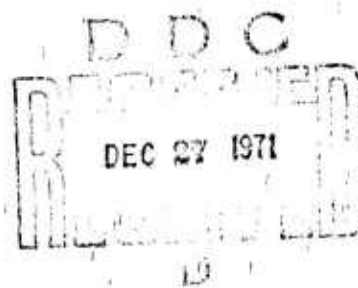# ON THE MAXIMUM OF A STATIONARY INDEPENDENT INCREMENT PROCESS

by

SHELDON M. ROSS

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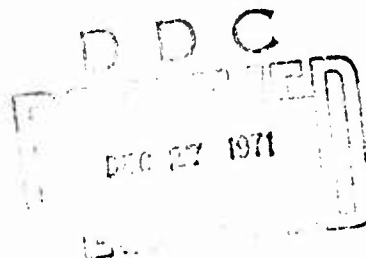
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Sheldon M. Ross
Department of Industrial Engineering
and Operations Research
University of California, Berkeley

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ABSTRACT

A stationary independent increment process is the continuous
time analogue of the discrete random walk, and, as such, has
a wide variety of applications. In this paper we consider
$M(t)$, the maximum value that such a process attains by time
t . By using renewal theoretic methods we obtain results
about $M(t)$. In particular we show that if μ , the mean
drift of the process, is positive, then $\dfrac{M(t)}{t}$ converges to
μ , and $E\bigl(M(t+h) - M(t)\bigr) \to h\mu$.

ON THE MAXIMUM OF A STATIONARY INDEPENDENT INCREMENT PROCESS

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Let $\{X(t) , t \geq 0\}$ be a separable stationary independent increment process continuous in probability and with $X(0) = 0$. Suppose that $E[X(t)] = \mu t$ where $\mu > 0$, and let

$$M(t) = \sup_{0 \leq s \leq t} X(s) .$$

Theorem 1:

    (i.) $M(t)/t \xrightarrow{a.s.} \mu$ as $t \rightarrow \infty$

    (ii) $EM(t)/t \rightarrow \mu$ as $t \rightarrow \infty$.

Proof:

Fix some constant $b > 0$, and define T_n, J_n, $n \geq 1$, by

$$T_1 = \inf \{t : X(t) \geq b\} , \quad X(T_1) = b + J_1$$

$$T_n = \inf \{t : X(t) \geq b + X(T_{n-1})\} , \quad X(T_n) = b + X(T_{n-1}) + J_n , \quad n > 1 .$$

Now, due to the stationary independent increment assumption, it follows that $\{T_n - T_{n-1} , n \geq 1\}$ is a renewal process. Letting $N(t)$ be the number of renewals by time t for this process, i.e., $N(t) = \max \{n : T_n \leq t\}$, we have

$$(1) \qquad bN(t) + J_1 + \ldots + J_{N(t)} \leq M(t) \leq bN(t) + J_1 + \ldots + J_{N(t)} + b .$$

Now, from the strong law of large numbers and the renewal result $N(t)/t \xrightarrow{a.s.} 1/ET_1$, we have that

$$(2) \qquad 1/t \ (bN(t) + J_1 + \ldots + J_{N(t)}) \xrightarrow{a.s.} \frac{b + EJ_1}{ET_1} \ .$$

Now, from a similar theorem in discrete time (i.e., Stein's lemma), it follows that $ET_1 < \infty$ and thus from Martingale theory

$$(3) \qquad E[b + J_1] = E[X(T_1)] = \mu ET_1 \ ,$$

and hence (i) follows from (1), (2) and (3). To prove (ii), we first note that

$$1/t \ E[bN(t) + J_1 + \ldots + J_{N(t)}] = bEN(t)/t + \frac{EJ_1(EN(t) + 1)}{t} - \frac{E[J_{N(t)+1}]}{t}$$

by Wald's equation. However by regarding J_1 as a reward earned at the time of the ith renewal it follows, since $EJ_1 < \infty$, by standard arguments ([1], Page 53) that $\dfrac{E[J_{N(t)+1}]}{t} \to 0$. Hence, by the elementary renewal theorem,

$$1/t \ E[bN(t) + J_1 + \ldots + J_{N(t)}] \to \frac{b + EJ_1}{ET_1} = \mu$$

and (ii) is proven.

Q.E.D.

Theorem 2:

If $E[X^2(t)] < \infty$, then, for each $h > 0$,

$$E[M(t + h) - M(t)] \to h\mu \ .$$

Proof:

By (1) we have

$$b(N(t + h) - N(t)) + J_1 + \ldots + J_{N(t+h)} - (J_1 + \ldots + J_{N(t)}) - b \le M(t + h) - $$

$$M(t) \le b(N(t + h) - N(t)) + J_1 + \ldots + J_{N(t+h)} - (J_1 + \ldots + J_{N(t)}) + b \ .$$

Taking expectations, and using Wald's equation, yields

$$(b + EJ_1)E[N(t + h) - N(t)] + E[J_{N(t)+1}] - E[J_{N(t+h)+1}] - b \leq$$

$$E[M(t + h) - M(t)] \leq (b + EJ_1)E[N(t + h) - N(t)] + E[J_{N(t)+1}] -$$

$$E[J_{N(t+h)+1}] + b .$$

We shall show in the ensuing lemma that $\lim\limits_{t \to \infty} E[J_{N(t)+1}]$ exists and is finite. Hence, the result follows from the above by first applying Blackwell's theorem and then letting $b \to 0$.

<div align="right">Q.E.D.</div>

__Lemma 1:__

If $E[X^2(t)] < \infty$, then

$$\lim_{t \to \infty} E[J_{N(t)+1}] = \frac{E[J_1 T_1]}{E[T_1]} < \infty .$$

__Proof:__

Let $g(t) = E[J_{N(t)+1}]$ and let F denote the distribution of T_1 . Then

$$g(t) = \int_0^\infty E[J_{N(t)+1} \mid T_1 = x]dF(x)$$

$$= \int_0^t g(t - x)dF(x) + \int_t^\infty E[J_1 \mid T_1 = x]dF(x) .$$

This renewal type equation has the solution

$$g(t) = h(t) + \int_0^t h(t - x)dm(x)$$

where

$$h(t) = \int_t^\infty E[J_1 \mid T_1 = x] dF(x) .$$

From the key renewal theorem it follows that

$$\lim_{t \to \infty} g(t) = \frac{\int_0^\infty h(t)dt}{E[T_1]} = \frac{\int_0^\infty \int_t^\infty E[J_1 \mid T_1 = x] dF(x) dt}{E[T_1]} =$$

$$\frac{\int_0^\infty x E[J_1 \mid T_1 = x] dF(x)}{E[T_1]} = \frac{E[J_1 T_1]}{E[T_1]}$$

where the interchange is justified by the nonnegativity of rewards. That $E[J_1 T_1] < \infty$ follows from a similar result in discrete time. (The discrete time result states that for a random walk with positive drift whose steps have finite variance, the variance of the value of the process the first time it exceeds b is finite. The equivalent discrete time version of $E[J_1 T_1] < \infty$ then follows from Stein's lemma upon application of the Cauchy-Schwarz inequality.)

$$\text{Q.E.D.}$$

Remarks:

The technique employed in the above proof is similar to the ladder variable approach in fluctuation theory. That is, we say that a ladder point is reached not when the process attains a new high but when it passes the previous ladder height by at least some fixed value. It is to be hoped that this technique will prove useful in studying other properties of stationary independent increment processes. Also this approach should be contrasted with the approach, recently developed by M. Rubinovitch, in which a point t is said to be a ladder point if $M(t) = X(t)$. By applying the theory of regenerative events, as developed by Kingman, Rubinovitch

is then able to obtain many interesting results.

For a discrete time proof of (i), the interested reader should consult Heyde.

REFERENCES

[1] Ross, S., APPLIED PROBABILITY MODELS WITH OPTIMIZATION APPLICATIONS, Holden-Day, (1970).

[2] Heyde, C.C., "Some Renewal Theorems with Application to a First Passage Problem," Annals of Mathematical Statistics, Vol. 37, pp. 699-711, (1966).

[3] Rubinovitch, M., "Ladder Regenerative Events with Applications to Dam Models," Technical Report No. 58, Department of Operations Research, Cornell University, (1968).